on the speed of the waves on the string and the wavelength of the waves. The six strings have different linear densities and are "tuned" by changing the tensions in the strings. We will see in **Interference of Waves** that the wavelength depends on the length of the strings and the boundary conditions. To play notes other than the fundamental notes, the lengths of the strings are changed by pressing down on the strings.

**16.5 Check Your Understanding** The wave speed of a wave on a string depends on the tension and the linear mass density. If the tension is doubled, what happens to the speed of the waves on the string?

## Speed of Compression Waves in a Fluid

The speed of a wave on a string depends on the square root of the tension divided by the mass per length, the linear density. In general, the speed of a wave through a medium depends on the elastic property of the medium and the inertial property of the medium.

$$|v| = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}}$$

The elastic property describes the tendency of the particles of the medium to return to their initial position when perturbed. The inertial property describes the tendency of the particle to resist changes in velocity.

The speed of a longitudinal wave through a liquid or gas depends on the density of the fluid and the bulk modulus of the fluid,

$$v = \sqrt{\frac{B}{\rho}}. \tag{16.9}$$

Here the bulk modulus is defined as $B = -\frac{\Delta P}{\frac{\Delta V}{V_0}}$, where $\Delta P$ is the change in the pressure and the denominator is the ratio of the change in volume to the initial volume, and $\rho \equiv \frac{m}{V}$ is the mass per unit volume. For example, sound is a mechanical wave that travels through a fluid or a solid. The speed of sound in air with an atmospheric pressure of $1.013 \times 10^5$ Pa and a temperature of $20°\text{C}$ is $v_\text{s} \approx 343.00\,\text{m/s}.$ Because the density depends on temperature, the speed of sound in air depends on the temperature of the air. This will be discussed in detail in **Sound**.

# 16.4 | Energy and Power of a Wave

### Learning Objectives

By the end of this section, you will be able to:

- Explain how energy travels with a pulse or wave
- Describe, using a mathematical expression, how the energy in a wave depends on the amplitude of the wave

All waves carry energy, and sometimes this can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls (**Figure 16.15**). Loud sounds can pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

**Figure 16.15** The destructive effect of an earthquake is observable evidence of the energy carried in these waves. The Richter scale rating of earthquakes is a logarithmic scale related to both their amplitude and the energy they carry.

In this section, we examine the quantitative expression of energy in waves. This will be of fundamental importance in later discussions of waves, from sound to light to quantum mechanics.

## Energy in Waves

The amount of energy in a wave is related to its amplitude and its frequency. Large-amplitude earthquakes produce large ground displacements. Loud sounds have high-pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. Consider the example of the seagull and the water wave earlier in the chapter (**Figure 16.3**). Work is done on the seagull by the wave as the seagull is moved up, changing its potential energy. The larger the amplitude, the higher the seagull is lifted by the wave and the larger the change in potential energy.

The energy of the wave depends on both the amplitude and the frequency. If the energy of each wavelength is considered to be a discrete packet of energy, a high-frequency wave will deliver more of these packets per unit time than a low-frequency wave. We will see that the average rate of energy transfer in mechanical waves is proportional to both the square of the amplitude and the square of the frequency. If two mechanical waves have equal amplitudes, but one wave has a frequency equal to twice the frequency of the other, the higher-frequency wave will have a rate of energy transfer a factor of four times as great as the rate of energy transfer of the lower-frequency wave. It should be noted that although the rate of energy transport is proportional to both the square of the amplitude and square of the frequency in mechanical waves, the rate of energy transfer in electromagnetic waves is proportional to the square of the amplitude, but independent of the frequency.

## Power in Waves

Consider a sinusoidal wave on a string that is produced by a string vibrator, as shown in **Figure 16.16**. The string vibrator is a device that vibrates a rod up and down. A string of uniform linear mass density is attached to the rod, and the rod oscillates the string, producing a sinusoidal wave. The rod does work on the string, producing energy that propagates along the string. Consider a mass element of the string with a mass $\Delta m$, as seen in **Figure 16.16**. As the energy propagates along the string, each mass element of the string is driven up and down at the same frequency as the wave. Each mass element of the string can be modeled as a simple harmonic oscillator. Since the string has a constant linear density $\mu = \frac{\Delta m}{\Delta x}$, each mass element of the string has the mass $\Delta m = \mu \Delta x$.

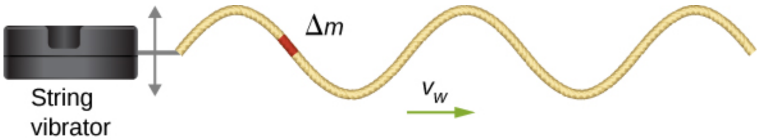


**Figure 16.16** A string vibrator is a device that vibrates a rod. A string is attached to the rod, and the rod does work on the string, driving the string up and down. This produces a sinusoidal wave in the string, which moves with a wave velocity *v*. The wave speed depends on the tension in the string and the linear mass density of the string. A section of the string with mass $\Delta m$ oscillates at the same frequency as the wave.

The total mechanical energy of the wave is the sum of its kinetic energy and potential energy. The kinetic energy $K = \frac{1}{2}mv^2$ of each mass element of the string of length $\Delta x$ is $\Delta K = \frac{1}{2}(\Delta m)v_y^2,$ as the mass element oscillates perpendicular to the direction of the motion of the wave. Using the constant linear mass density, the kinetic energy of each mass element of the string with length $\Delta x$ is

$$\Delta K = \frac{1}{2}(\mu \Delta x)v_y^2.$$

A differential equation can be formed by letting the length of the mass element of the string approach zero,

$$dK = \lim_{\Delta x \to 0} \frac{1}{2}(\mu \Delta x)v_y^2 = \frac{1}{2}(\mu dx)v_y^2.$$

Since the wave is a sinusoidal wave with an angular frequency $\omega,$ the position of each mass element may be modeled as $y(x,\, t) = A\sin(kx - \omega t).$ Each mass element of the string oscillates with a velocity $v_y = \frac{\partial y(x,\, t)}{\partial t} = -A\omega\cos(kx - \omega t).$ The kinetic energy of each mass element of the string becomes

$$\begin{aligned} dK &= \frac{1}{2}(\mu dx)(-A\omega\cos(kx - \omega t))^2, \\ &= \frac{1}{2}(\mu dx)A^2\omega^2\cos^2(kx - \omega t). \end{aligned}$$

The wave can be very long, consisting of many wavelengths. To standardize the energy, consider the kinetic energy associated with a wavelength of the wave. This kinetic energy can be integrated over the wavelength to find the energy associated with each wavelength of the wave:

$$\begin{aligned} dK &= \frac{1}{2}(\mu dx)A^2\omega^2\cos^2(kx), \\ \int_0^{K_\lambda} dK &= \int_0^{\lambda} \frac{1}{2}\mu A^2\omega^2\cos^2(kx)dx = \frac{1}{2}\mu A^2\omega^2 \int_0^{\lambda}\cos^2(kx)dx, \\ K_\lambda &= \frac{1}{2}\mu A^2\omega^2\left[\frac{1}{2}x + \frac{1}{4k}\sin(2kx)\right]_0^{\lambda} = \frac{1}{2}\mu A^2\omega^2\left[\frac{1}{2}\lambda + \frac{1}{4k}\sin(2k\lambda) - \frac{1}{4k}\sin(0)\right], \\ K_\lambda &= \frac{1}{4}\mu A^2\omega^2\lambda. \end{aligned}$$

There is also potential energy associated with the wave. Much like the mass oscillating on a spring, there is a conservative restoring force that, when the mass element is displaced from the equilibrium position, drives the mass element back to the equilibrium position. The potential energy of the mass element can be found by considering the linear restoring force of the string, In **Oscillations**, we saw that the potential energy stored in a spring with a linear restoring force is equal to $U = \frac{1}{2}k_s x^2,$ where the equilibrium position is defined as $x = 0.00\text{ m}.$ When a mass attached to the spring oscillates in simple harmonic motion, the angular frequency is equal to $\omega = \sqrt{\frac{k_s}{m}}.$ As each mass element oscillates in simple harmonic motion, the spring constant is equal to $k_s = \Delta m\omega^2.$ The potential energy of the mass element is equal to

$$\Delta U = \frac{1}{2}k_s x^2 = \frac{1}{2}\Delta m\omega^2 x^2.$$

Note that $k_s$ is the spring constant and not the wave number $k = \frac{2\pi}{\lambda}$. This equation can be used to find the energy over a wavelength. Integrating over the wavelength, we can compute the potential energy over a wavelength:

$$\begin{aligned} dU &= \frac{1}{2}k_s x^2 = \frac{1}{2}\mu\omega^2 x^2 dx, \\ U_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \cos^2(kx)dx = \frac{1}{4}\mu A^2 \omega^2 \lambda. \end{aligned}$$

The potential energy associated with a wavelength of the wave is equal to the kinetic energy associated with a wavelength. The total energy associated with a wavelength is the sum of the potential energy and the kinetic energy:

$$\begin{aligned} &E_\lambda = U_\lambda + K_\lambda, \\ &E_\lambda = \frac{1}{4}\mu A^2 \omega^2 \lambda + \frac{1}{4}\mu A^2 \omega^2 \lambda = \frac{1}{2}\mu A^2 \omega^2 \lambda. \end{aligned}$$

The time-averaged power of a sinusoidal mechanical wave, which is the average rate of energy transfer associated with a wave as it passes a point, can be found by taking the total energy associated with the wave divided by the time it takes to transfer the energy. If the velocity of the sinusoidal wave is constant, the time for one wavelength to pass by a point is equal to the period of the wave, which is also constant. For a sinusoidal mechanical wave, the time-averaged power is therefore the energy associated with a wavelength divided by the period of the wave. The wavelength of the wave divided by the period is equal to the velocity of the wave,

$$P_{\text{ave}} = \frac{E_\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 \frac{\lambda}{T} = \frac{1}{2}\mu A^2 \omega^2 v. \tag{16.10}$$

Note that this equation for the time-averaged power of a sinusoidal mechanical wave shows that the power is proportional to the square of the amplitude of the wave and to the square of the angular frequency of the wave. Recall that the angular frequency is equal to $\omega = 2\pi f$, so the power of a mechanical wave is equal to the square of the amplitude and the square of the frequency of the wave.

## Example 16.6

### Power Supplied by a String Vibrator

Consider a two-meter-long string with a mass of 70.00 g attached to a string vibrator as illustrated in **Figure 16.16**. The tension in the string is 90.0 N. When the string vibrator is turned on, it oscillates with a frequency of 60 Hz and produces a sinusoidal wave on the string with an amplitude of 4.00 cm and a constant wave speed. What is the time-averaged power supplied to the wave by the string vibrator?

### Strategy

The power supplied to the wave should equal the time-averaged power of the wave on the string. We know the mass of the string $(m_s)$, the length of the string $(L_s)$, and the tension $(F_T)$ in the string. The speed of the wave on the string can be derived from the linear mass density and the tension. The string oscillates with the same frequency as the string vibrator, from which we can find the angular frequency.

### Solution

1. Begin with the equation of the time-averaged power of a sinusoidal wave on a string:

$$P = \frac{1}{2}\mu A^2 \omega^2 v.$$

   The amplitude is given, so we need to calculate the linear mass density of the string, the angular frequency of the wave on the string, and the speed of the wave on the string.

2. We need to calculate the linear density to find the wave speed:

$$\mu = \frac{m_s}{L_s} = \frac{0.070 \text{ kg}}{2.00 \text{ m}} = 0.035 \text{ kg/m}.$$

3. The wave speed can be found using the linear mass density and the tension of the string:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{90.00 \text{ N}}{0.035 \text{ kg/m}}} = 50.71 \text{ m/s}.$$

4. The angular frequency can be found from the frequency:

$$\omega = 2\pi f = 2\pi\left(60 \text{ s}^{-1}\right) = 376.80 \text{ s}^{-1}.$$

5. Calculate the time-averaged power:

$$P = \frac{1}{2}\mu A^2 \omega^2 v = \frac{1}{2}\left(0.035\frac{\text{kg}}{\text{m}}\right)(0.040 \text{ m})^2 \left(376.80 \text{ s}^{-1}\right)^2 \left(50.71\frac{\text{m}}{\text{s}}\right) = 201.59 \text{ W}.$$

### Significance

The time-averaged power of a sinusoidal wave is proportional to the square of the amplitude of the wave and the square of the angular frequency of the wave. This is true for most mechanical waves. If either the angular frequency or the amplitude of the wave were doubled, the power would increase by a factor of four. The time-averaged power of the wave on a string is also proportional to the speed of the sinusoidal wave on the string. If the speed were doubled, by increasing the tension by a factor of four, the power would also be doubled.

**16.6 Check Your Understanding** Is the time-averaged power of a sinusoidal wave on a string proportional to the linear density of the string?

The equations for the energy of the wave and the time-averaged power were derived for a sinusoidal wave on a string. In general, the energy of a mechanical wave and the power are proportional to the amplitude squared and to the angular frequency squared (and therefore the frequency squared).

Another important characteristic of waves is the intensity of the waves. Waves can also be concentrated or spread out. Waves from an earthquake, for example, spread out over a larger area as they move away from a source, so they do less damage the farther they get from the source. Changing the area the waves cover has important effects. All these pertinent factors are included in the definition of **intensity (*I*)** as power per unit area:

$$I = \frac{P}{A}, \tag{16.11}$$

where $P$ is the power carried by the wave through area $A$. The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter ($\text{W/m}^2$). Many waves are spherical waves that move out from a source as a sphere. For example, a sound speaker mounted on a post above the ground may produce sound waves that move away from the source as a spherical wave. Sound waves are discussed in more detail in the next chapter, but in general, the farther you are from the speaker, the less intense the sound you hear. As a spherical wave moves out from a source, the surface area of the wave increases as the radius increases $\left(A = 4\pi r^2\right)$. The intensity for a spherical wave is therefore

$$I = \frac{P}{4\pi r^2}. \tag{16.12}$$

If there are no dissipative forces, the energy will remain constant as the spherical wave moves away from the source, but the intensity will decrease as the surface area increases.

In the case of the two-dimensional circular wave, the wave moves out, increasing the circumference of the wave as the radius of the circle increases. If you toss a pebble in a pond, the surface ripple moves out as a circular wave. As the ripple moves away from the source, the amplitude decreases. The energy of the wave spreads around a larger circumference and the amplitude decreases proportional to $\frac{1}{r}$, which is also the same in the case of a spherical wave, since intensity is proportional to the amplitude squared.

# 16.5 | Interference of Waves

## Learning Objectives

By the end of this section, you will be able to:

- Explain how mechanical waves are reflected and transmitted at the boundaries of a medium
- Define the terms interference and superposition
- Find the resultant wave of two identical sinusoidal waves that differ only by a phase shift

Up to now, we have been studying mechanical waves that propagate continuously through a medium, but we have not discussed what happens when waves encounter the boundary of the medium or what happens when a wave encounters another wave propagating through the same medium. Waves do interact with boundaries of the medium, and all or part of the wave can be reflected. For example, when you stand some distance from a rigid cliff face and yell, you can hear the sound waves reflect off the rigid surface as an echo. Waves can also interact with other waves propagating in the same medium. If you throw two rocks into a pond some distance from one another, the circular ripples that result from the two stones seem to pass through one another as they propagate out from where the stones entered the water. This phenomenon is known as interference. In this section, we examine what happens to waves encountering a boundary of a medium or another wave propagating in the same medium. We will see that their behavior is quite different from the behavior of particles and rigid bodies. Later, when we study modern physics, we will see that only at the scale of atoms do we see similarities in the properties of waves and particles.

## Reflection and Transmission

When a wave propagates through a medium, it reflects when it encounters the boundary of the medium. The wave before hitting the boundary is known as the incident wave. The wave after encountering the boundary is known as the reflected wave. How the wave is reflected at the boundary of the medium depends on the boundary conditions; waves will react differently if the boundary of the medium is fixed in place or free to move (Figure 16.17). A **fixed boundary condition** exists when the medium at a boundary is fixed in place so it cannot move. A **free boundary condition** exists when the medium at the boundary is free to move.